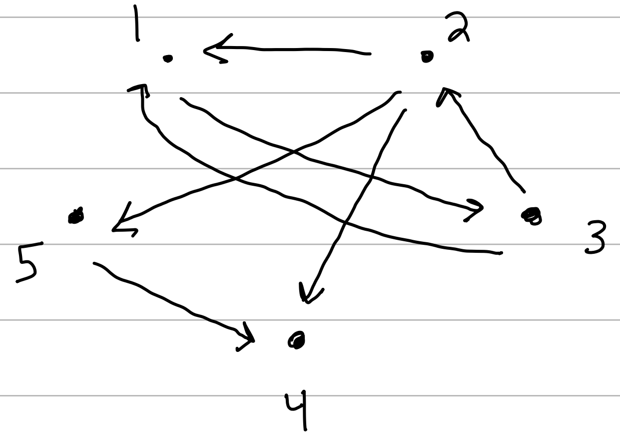
9/5/25 Lecture 6

Matrix Transformations

$A = m \times n$ matrix gives us a fn $T = T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by
$T(\vec{x}) = A\vec{x}$

Page Rank Example



- arrow indicates a page linking to another

5x5 Matrix $\tilde{A} = (\tilde{a}_{ij})$ where

$$\tilde{a}_{ij} = \begin{cases} 0 & \text{page } j \text{ \underline{does not} link to } i \\ 1 & \text{where page } j \text{ links to page } i \end{cases}$$

- Then scale the cols to get a matrix A whose cols all sum to 1.

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}$$

vector1 : [1000 , 0 , 0 , 0 , 0] → initial vector

* Starting 1000 people on page 1

If you multiply vector1 by A then you see where people end up.

Keep doing this and then you get $\begin{bmatrix} 249 \\ 186 \\ 376 \\ 124 \\ 63 \end{bmatrix}$

- Higher number indicates importance of web page.

Fibonacci Sequence.

Pick 2 starting numbers : $x_0 = 1$, $x_1 = 1$

- Then keep adding previous two numbers to get next number

Alt: Let $\vec{x}_1 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$, $\vec{x}_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{x}_3 \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$

$$x_n = \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}$$

$$\vec{X}_{n+1} = \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ x_{n-1} + x_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \vec{x}_n$$

Q: How to know whether or not a given function fn $T: \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation?

e.g. $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_2\\x_1+x_2\end{pmatrix} = \begin{pmatrix}0&1\\1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$ $\quad T: \mathbb{R}^2 \to \mathbb{R}^2$

e.g. $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \,,\begin{pmatrix}1\\2\\0\end{pmatrix} + x_2\begin{pmatrix}-1\\0\\3\end{pmatrix} = \begin{bmatrix}1&-1\\2&0\\0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

e.g.1. $T: \mathbb{R}^2 \to \mathbb{R}^3$ $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1-2\\x_2+1\\2x_1+x_2\end{pmatrix}$

* Not a matrix transformation

e.g.2 $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T(\vec{v}) = \vec{v}$ rotated counterclockwise by $\frac{\pi}{3}$ radians

* Don't know yet

Def - A fn $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if:

(i) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

(ii) $T(c\vec{x}) = cT(\vec{x})$

Prop says - Matrix transformations are linear transformations

* does not mean linear transformations are matrix transformations

e.g. 1

Checking (i):

$$T\left(\begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}\right) = T\begin{pmatrix}7\\3\end{pmatrix} = \begin{pmatrix}5\\4\\17\end{pmatrix}$$

$\neq$ So this is not a linear transformation

$$T\left(\begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}\right) = \begin{pmatrix}1\\2\\7\end{pmatrix} + \begin{pmatrix}2\\3\\10\end{pmatrix} = \begin{pmatrix}3\\5\\17\end{pmatrix}$$

T: is a linear transformation

T: is not a matrix transformation

I think this is wrong

Theorem: Any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation w matrix

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_n) \end{bmatrix}$$

and $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

We will pick up from here next lecture